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Free electromagnetic potentials in Minkowski and Euclidean regions

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Abstract. Free electromagnetic vector potentials in Coulomb and Gupta–Bleuler gauges are shown to be unitary equivalent in both Minkowski and Euclidean regions. For covariant gauges, the Euclidean electromagnetic potential is Markovian but non-reflective, whereas for the Coulomb gauge it is reflective but only satisfies the Markov property with respect to special half-spaces. The Feynman–Kac–Nelson formula can be established for the case of the Coulomb gauge.

1. Introduction

The problems related to gauges in quantum electrodynamics have been discussed extensively by Strocchi and Wightman (1974). Their analysis indicated that it is impossible to formulate a relativistic covariant and local (or weakly local) theory of quantum electrodynamics in a positive metric Hilbert space using only the four-vector potential A_μ. To overcome this difficulty one can either introduce a Hilbert space with indefinite metric, or one has to give up locality and relativistic covariance in order to retain the positive metric Hilbert space.

The first method is known as the Gupta–Bleuler formulation which involves three Hilbert spaces $\mathscr{H}'' \subset \mathscr{H}' \subset \mathscr{H}$. A sesquilinear Hermitian form $\langle\, .\, ,\, .\,\rangle$ exists which is indefinite on $\mathscr{H}$, non-negative on $\mathscr{H}'$ and zero on $\mathscr{H}''$. The closed subspace $\mathscr{H}'$ of $\mathscr{H}$ is spanned by the states $|\psi\rangle$ satisfying the nonlinear Gupta–Bleuler subsidiary condition

$$\partial^\mu A_\mu^{(-)}(x)|\psi\rangle = 0.$$

The physical space for the photon, denoted by $\mathscr{H}_{\text{phys}}$, is then given by the quotient space $\mathscr{H}'/\mathscr{H}''$. Since $\mathscr{H}'$ is not dense in $\mathscr{H}$, the Maxwell equations only hold as a mean value between physical states.

The second approach is known as the Coulomb or radiation gauge formulation and is specified by the gauge conditions $\vec{\nabla}\,.\,\vec{A}(x) = 0$, $A_{.}(x) = 0$. In this gauge $\mathscr{H} = \mathscr{H}' = \mathscr{H}_{\text{phys}}$ and $\mathscr{H}'' = 0$; $\langle\, .\, ,\, .\,\rangle$ is positive definite on $\mathscr{H}'$. Now the Maxwell equations hold as operator equations. However, this formulation is no longer manifestly covariant and local, so it is necessary to supply a Lorentz transformation with a gauge-dependent term in order to obtain covariance of the Coulomb gauge.

Despite the differences in their mathematical structures, the physical contents of both the covariant Gupta–Bleuler gauge formulation and the non-covariant Coulomb gauge formulation are the same, in particular the transition probabilities. Ferrari *et al* (1974) have shown the equivalence of the theories derived from A^{C} and A^{F}, the vector

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potentials in the Coulomb and Feynman gauges respectively, in terms of their Wightman functions. In this paper we shall show that there exists a unitary map connecting the Gupta–Bleuler gauge and Coulomb gauge formulations in both Minkowski and Euclidean regions. We shall also extend the results for Euclidean electromagnetic vector potentials in covariant gauges (Guerra 1976, Lim 1976) to the case of the Coulomb gauge. It is found that the Euclidean vector potential in the Coulomb gauge is reflective, but it only satisfies the Markov property with respect to the half-spaces bounded by hyperplanes $x_4 = $ constant. Finally the Feynman–Kac–Nelson formula for such a field can be established in the usual manner.

2. Free electromagnetic potential in the Minkowski region

The relativistic two-point function for an electromagnetic potential in an arbitrary gauge can be expressed in the form

$$W_{\mu\nu}(x-y) = \langle\psi_0, A_\mu(x)A_\nu(y)\psi_0\rangle$$
$$= \frac{1}{(2\pi)^3}\int D_{\mu\nu}(p)\exp[\mathrm{i}|\vec{p}|(x_0-y_0)]\exp[\mathrm{i}\vec{p}\,.\,(\vec{x}-\vec{y})]\frac{\mathrm{d}^3p}{2|\vec{p}|} \tag{2.1a}$$

where†

$$D_{\mu\nu}(p) = -g_{\mu\nu} + p_\mu p_\nu d_1(p^2, (n\,.\,p)^2)$$
$$+ p^2 n_\mu n_\nu d_2(p^2, (n\,.\,p)^2 - (n\,.\,p)(n_\mu p_\nu + n_\nu p_\mu)d_3(p^2, (n\,.\,p)^2); \tag{2.1b}$$

n is a unit time-like vector and d_1, d_2 and d_3 are gauge-dependent functions of p^2 and $(n\,.\,p)^2$.

There are two important cases to be considered, namely

(i) covariant gauges with $D_{\mu\nu}(p)$ independent of n, i.e. $d_1 = d_1(p^2)$ and $d_2 = 0 = d_3$ and

(ii) non-covariant gauges with at least one of the non-covariants in $D_{\mu\nu}(p)$ (i.e. terms that depend on n) not vanishing.

A special case of (i) is the one-parameter family of Gupta–Bleuler gauges characterised by

$$D^{\mathrm{G}}_{\mu\nu}(p) = -g_{\mu\nu} - (\alpha-1)p_\mu p_\nu/p^2 \tag{2.2}$$

where α is a real parameter. The values of α corresponding to Feynman, Landau and Fried–Yennie gauges are respectively 1, 0 and 3.

As for (ii), a well known example is the radiation or Coulomb gauge which is specified by $d_1 = d_2 = d_3 = [(n\,.\,p)^2 - p^2]^{-1}$. If n is chosen to be (1, 0, 0, 0) then $D_{\mu\nu} = 0$ for $\mu = 0, 1, 2, 3$, and the corresponding expression for $D_{\mu\nu}(p)$ in the Coulomb gauge becomes

$$D^{\mathrm{C}}_{ij}(p) = \left(\delta_{ij} - \frac{p_i p_j}{p^2}\right). \tag{2.3}$$

The relativistic one-particle space for the photon in the Gupta–Bleuler gauge can be

† We have taken the Minkowski metric $g_{\mu\nu}$ to be $g_{\mu 0} = +\delta_{\mu 0}$, $g_{ij} = -\delta_{ij}$, and $n\,.\,p = n_0p_0 - \vec{n}\,.\,\vec{p}$.

defined as follows. Let $\mathscr{H}$ be the completion of the vector-valued Schwartz test-function space $\mathscr{S}(\mathbb{R}^4)\times\mathbb{C}^4$ with respect to the inner product

$$\langle f, g\rangle_{\mathscr{H}} = \sum_{\mu,\nu}\int\int \overline{f^{\mu}(x)}W^{G}_{\mu\nu}(x-y)g^{\nu}(y)\,d^4x\,d^4y; \tag{2.4}$$

then $\mathscr{H}$ is an indefinite metric Hilbert space since $W_{\mu\nu}$ is indefinite. We remark that this property is necessary for one to obtain a unitary representation of the Poincare group.

By imposing the Gupta–Bleuler condition on $\mathscr{H}$ we get a positive semidefinite closed subspace

$$\mathscr{H}' = \{f\in\mathscr{H}|p\,.\,\tilde{f}=0 \text{ almost everywhere on } C_+\}$$

where $\tilde{f}$ is the Fourier transform of f and C_+ is the mantle of the forward light cone. Now $\mathscr{H}'$ is independent of the gauge parameter α. The physical one-photon space in the Gupta–Bleuler gauge is then given by the quotient space $\mathscr{H}^G = \mathscr{H}'/\mathscr{H}''$, where $\mathscr{H}''$ is the subspace with vanishing norm.

The one-particle space for the photon in the Coulomb gauge can be defined as

$$\mathscr{H}^C = \mathscr{H}'\cap\mathscr{H}^0$$

where $\mathscr{H}^0 \equiv \{f\in\mathscr{H}'|f^0=0\}$. Clearly, $\mathscr{H}^C$ is a Hilbert space with positive metric.

The physical equivalence of these two formulations can be shown in terms of their one-particle spaces.

Proposition 1. There exists a unitary map

$$\gamma: \hat{f}^{\mu}\to\tilde{f}^{\mu}-p^{\mu}(\tilde{f}^0/p^0) \tag{2.5}$$

which defines a unitary equivalence $\mathscr{H}^C\cong\mathscr{H}^G$.

Proof. First we note that

$$(\gamma\tilde{f})^0=0 \qquad \forall f\in\mathscr{H}'.$$

Therefore $\mathscr{H}^C=\gamma\mathscr{H}'$. But $(I-\gamma)$ maps $\mathscr{H}'$ into $\mathscr{H}''$, which is spanned by vectors of the form $\tilde{h}^{\mu}=p^{\mu}\lambda(p)$, where $\lambda(p)\in\mathscr{S}(\mathbb{R}^4)$ is a arbitrary Schwartz test function. γ vanishes on $\mathscr{H}''$ and $\gamma\tilde{h}=0$ implies $\tilde{h}\in\mathscr{H}''$. Hence $\mathscr{H}''$ is the kernel of γ. Furthermore, γ is well defined as can be seen by restricting the Taylor expansion about $p=0$ to V_+. Thus

$$\tilde{f}\to\tilde{f}+(I-\gamma)\tilde{f}$$

defines a unique decomposition

$$\mathscr{H}'=\mathscr{H}^C\oplus\mathscr{H}''$$

or

$$\mathscr{H}^C\cong\mathscr{H}^C=\mathscr{H}''/\mathscr{H}''. \quad \square$$

The above results can be generalised to the case of field algebra as defined by Borchers (1962). Let $\mathscr{S}_n$ be the n-fold tensor product space of $\mathscr{S}(\mathbb{R}^4)\times\mathbb{R}^4$, with $\mathscr{S}_0=\mathbb{C}$. Denote by $\mathfrak{N}$ the locally convex direct sum of these spaces, i.e. $\mathfrak{N}=\oplus_{n=0}^{\infty}\mathscr{S}_n$, which has elements in the form of a finite sequence $\mathfrak{N}\ni f=\{f_0,f_1,f_2,\ldots\}$, where $f_0\in\mathbb{C}$, $(f)_n\equiv f_n\in\mathscr{S}_n$. Equipping $\mathfrak{N}$ with the product

$$(f\times g)_n(x_1,\ldots,x_n)=\sum_{j=0}^{n}f_j(x_1,\ldots,x_j)g_{n-j}(x_{j+1},\ldots,x_n),$$

and the involution * defined by

$$(f^*)_n(x_1,\ldots,x_n)=\overline{f_n(x_n,\ldots,x_1)},$$

where —— denotes the complex conjugate, then $\mathfrak{N}$ becomes a *- test-function algebra (or Borchers algebra) for the photon potential.

In order for the two-point function of the electromagnetic potential in the Gupta–Bleuler gauge W^{G} to define a positive linear functional on $\mathfrak{N}$, the following transversality condition is necessary:

$$(p\,.\,\tilde{f})_n=(p_l)_{\mu_l}\tilde{f}_n^{\mu_1\ldots\mu_l\ldots\mu_n}=0 \qquad p_l\in V^+ \qquad l=1,\ldots,n. \tag{2.6}$$

Let $\mathfrak{N}_1$ be the test-function algebra satisfying condition (2.3). The two-sided ideal of $\mathfrak{N}_1$ is contained in the kernel of W^{G} and is given by

$$\mathscr{I}=\mathfrak{N}_1\cap\{\tilde{f}|(\tilde{f})_0\}=0,\ (\tilde{f})_n=\tilde{f}_n^{\mu_1\ldots\mu_l\ldots\mu_n}(p_1,\ldots,p_l,\ldots,p_n)$$
$$=p^{\mu_l}\tilde{\lambda}_{n-1}^{\mu_1\ldots\hat{\mu}_l\ldots\mu_n}(p_1,\ldots,\tilde{p}_l,\ldots,p_n)\ \text{for at least one } l$$

where $\lambda_{n-1}^{\mu_1\ldots\hat{\mu}_l\ldots\mu_n}\in\mathscr{S}_{n-1}$ is arbitrary. The physical test-function algebra for the photon in the Gupta–Bleuler gauge is given by the quotient algebra $\mathfrak{N}^{\mathrm{G}}\equiv\mathfrak{N}_1/\mathscr{I}$. The positive linear functional on $\mathfrak{N}^{\mathrm{G}}$ then determines a unique theory for covariant electromagnetic potentials through the Gelfand–Naimark–Segal construction.

The corresponding test-function algebra for the photon potential in the Coulomb gauge is given by

$$\mathfrak{N}^{\mathrm{C}}=\mathfrak{N}_1\cap\{f_n^{\mu_1\ldots\mu_n}=0 \text{ if any } \mu_l=0\}.$$

Let Γ be the natural algebraic generalisation of the map defined in proposition 1, with

$$(\Gamma f)_0=f_0 \qquad (\Gamma f)_n=\gamma\times\ldots\times\gamma f_n.$$

Then we have $\mathfrak{N}^{\mathrm{C}}=\mathrm{range}\ (\Gamma\mathfrak{N}_1)$. It is not difficult to see that the generalisation of proposition 1 holds for photon field algebras.

Proposition 2. Γ defines a *-algebra isomorphism

$$\mathfrak{N}^{\mathrm{C}}\cong\mathfrak{N}^{\mathrm{G}}=\mathfrak{N}_1/\mathscr{I}.$$

3. Free electromagnetic potentials in the Euclidean region

Euclidean electrodynamics was first studied by Schwinger (1959) and Fradkin (1967). Gross (1975) was the first to construct a free Euclidean electromagnetic potential in the same spirit as Nelson (1973). The properties of free vector potentials in various covariant gauges were studied independently by Guerra (1976) and Lim (1976). In this section we shall extend the results to the case of the non-covariant Coulomb gauge.

First consider the Euclidean (or Schwinger) two-point function of the electromagnetic potential in the covariant Gupta–Bleuler gauge:

$$S_{ij}^{\mathrm{G}}(\boldsymbol{x}-\boldsymbol{y})=\sum_{\mu,\nu}B_{i\mu}B_{j\nu}W^{\mathrm{G}}((\vec{x}-\vec{y}),\mathrm{i}(x_0-y_0))$$
$$=\frac{1}{(2\pi)^4}\int\left(\delta_{ij}+(\alpha-1)\frac{p_ip_j}{\boldsymbol{p}^2}\right)\exp[\mathrm{i}\boldsymbol{p}\,.\,(\boldsymbol{x}-\boldsymbol{y})]\,d^4p/\boldsymbol{p}^2 \tag{3.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ are the Euclidean four-vectors with $\boldsymbol{p}^2 = \Sigma_{i=1}^4 p_i^2$, and $\boldsymbol{p}\cdot\boldsymbol{x} = \Sigma_{i=1}^4 p_i x_i$, and $B_{i\mu} = 1$ if $i = \mu = 1, 2, 3$; $B_{40} = \mathrm{i}$ and $B_{i\mu} = 0$ otherwise. The matrix transformation $B_{i\mu}$ is required to change the Minkowski metric $g_{\mu\nu}$ to the Euclidean metric δ_{ij}. Actually one can also consider S_{ij}^{G} as the anti-time-ordered product of the four-vector potential $(\mathrm{i}A_0, \boldsymbol{A})$ in the Gupta–Bleuler gauge continued to imaginary time. In contrast to the relativistic two-point function, the Schwinger function is positive semidefinite if the real parameter α is non-negative. A more general form of S_{ij}^{G} can be obtained by replacing α by a non-negative measurable function $\alpha(\boldsymbol{p}^2)$.

The Euclidean vector potential in the Gupta–Bleuler gauge, denoted by $\mathscr{A}_i^{\mathrm{G}}$, is then defined as the real generalised gaussian random vector field with mean zero and covariance given by S_{ij}^{G}. This Euclidean field satisfies Nelson's Markov property provided the gauge function $\alpha(\boldsymbol{p}^2)$ has an inverse in the form of a polynomial in $\boldsymbol{p}^2$. The Markov property also holds for the case of the Landau gauge (with $\alpha(\boldsymbol{p}^2) = 0$) even though S_{ij}^{G} is singular in this case. However, for all these covariant gauges the Euclidean vector potentials fail to satisfy the reflection property. This is closely related to the existence of non-physical states in the relativistic region (see Lim 1976).

The free Euclidean electromagnetic field $\mathscr{F}_{ij}$ can be obtained from $\mathscr{A}_i^{\mathrm{G}}$ by the following relation:

$$\mathscr{F}_{ij} = \partial_i \mathscr{A}_j^{\mathrm{G}} - \partial_j \mathscr{A}_i^{\mathrm{G}}. \tag{3.2}$$

It has been shown (Yao 1976) that $\mathscr{F}_{ij}$ is both Markovian and reflective and that it leads to a Wightman theory in the relativistic region as expected. Here we note that the curl operation on $\mathscr{A}_i^{\mathrm{G}}$ preserves the Markov property of the field, but the local structures with the Markov property for $\mathscr{A}_i^{\mathrm{G}}$ and $\mathscr{F}_{ij}$ are different. In general the underlying probability space for $\mathscr{A}_i^{\mathrm{G}}$ is larger than that for $\mathscr{F}_{ij}$ because all the elements of the σ algebra generated by $\mathscr{F}_{ij}$ correspond to physical observables, whereas the σ algebra generated by $\mathscr{A}_i^{\mathrm{G}}$ includes measurable functions which do not correspond to physical observables.

Now consider the Euclidean one-particle space $\mathscr{K}$ given by the completion of the real vector-valued test-function space $\mathscr{S}(\mathbb{R}^4)\times\mathbb{R}^4$ with respect to the inner product

$$\langle f, g\rangle_{\mathscr{K}} = \sum_{i,j} \iint f_i(\boldsymbol{x}) S_{ij}(\boldsymbol{x}-\boldsymbol{y}) g_j(\boldsymbol{y})\, \mathrm{d}^4x\, \mathrm{d}^4y. \tag{3.3}$$

$\mathscr{K}$ has a non-negative metric in contrast to the indefinite metric of $\mathscr{H}$ in the relativistic case. In fact, the Euclidean theory is free from the difficulties arising from the gauge problem that exists in relativistic vector potentials, and there is no conflict between Euclidean covariance and Markovian local structure on the one hand, and the positive metric of the one-particle space on the other.

A natural question now arises. Is it possible to find a subspace of $\mathscr{K}$ such that both the Markov and reflection properties are satisfied? Our attempts to answer this question lead us to consider the Euclidean theory of the electromagnetic potential in the Coulomb gauge.

Consider the subspace $\mathscr{K}' \subset \mathscr{K}$ with elements satisfying the Euclidean transversality condition $\Sigma_{i=1}^4 \partial_i f_i = 0$. For a given choice of time axis it is possible to find a unique horizontal representative such that $f_4 = 0$. This can be achieved by the projection $f \to f_i - (p_i f_4/p_4)$. Denote by $\mathscr{K}^{\mathrm{C}}$ the closed subspace spanned by all such vectors (i.e. all $f \in \mathscr{K}$ such that $\Sigma_{i=1}^4 \partial_i f_i = 0$ and $f_4 = 0$). The space $\mathscr{K}'$ has the decomposition

$$\mathscr{K}' = \mathscr{K}^{\mathrm{C}} \oplus \mathscr{K}'' \tag{3.4}$$

where $\mathscr{K}''$ is the subspace of the longitudinal elements. However, such a decomposition is not Euclidean covariant since $\mathscr{H}^C$ is not left invariant under the full Euclidean group ISO(4).

The space $\mathscr{H}^C$ is none other than the one-particle space for $\mathscr{A}_i^C$, the Euclidean vector potential in the Coulomb gauge. It is clear from the above discussion that $\mathscr{A}_i^C$ is not Euclidean covariant. However, $\mathscr{A}_i^C$ satisfies the reflection property. Let θ be the unitary time-reflection operator defined by

$$\theta : f_i(x, x_4) \to (-1)^{\delta_{i4}} f_i(x, -x_4). \tag{3.5}$$

Since $f_4 = 0$ in $\mathscr{H}^C$, it is not difficult to show that $\mathscr{H}^C$ is invariant under θ. We then have the following proposition.

Proposition 3. The Euclidean electromagnetic potential in the Coulomb gauge satisfies the reflection property.

This result agrees with our remark that the reflection property is closely related to the metric of the relativistic Hilbert space of states, which is positive definite in this case. Thus $\mathscr{H}^C$ can be considered as the 'Euclidean physical subspace' in this sense.

In contrast to the Euclidean potentials in covariant gauges, the non-covariant $\mathscr{A}_i^C$ does not satisfy Nelson's Markov property. However, it is Markovian with respect to special half-spaces. Hegerfeldt (1974) has replaced Nelson's Markov property by a more general concept, the T positivity, such that it is still strong enough to allow the relativistic theory by analytic continuation. To recall the meaning of T positivity, let $E_\pm$ and E_0 denote the projections of $\mathscr{H}'$ with supports $\mathbb{R}^4_\pm = \{x \in \mathbb{R}^4 | x_4 \gtrless 0\}$ and $\mathbb{R}^4_0 = \{x \in \mathbb{R}^4 | x_4 = 0\}$ respectively, and the corresponding projections on $\mathscr{H}^C$ by $E^C_\pm$ and E^C_0. Then following Challifour (1976) T positivity means $T \equiv E_+ \theta E_+ \geqslant 0$. This is, in fact, a generalisation of Hegerfeldt's original definition which is in terms of expectational functionals, because $\mathscr{H}'_+$ is in general a proper subspace of the closed linear span $\overline{\mathrm{Sp}}\{\exp(\mathrm{i}\mathscr{A}(f)) | f \in \mathscr{H}'_+\}$. Now the $\mathscr{A}_i^C$ furnish an example of an Euclidean field which is T positive but yet fails to satisfy Nelson's Markov property.

Proposition 4. $\mathscr{A}_i^C$ is Markovian with respect to the half-spaces bounded by $x_4 =$ constant.

Proof. Basically our proof is based on the results of Hegerfeldt and Challifour. Consider the subspace $\mathscr{H}'_+ = E_+\mathscr{H}'$. According to the decomposition (3.4) any $f \in \mathscr{H}'_+$ can be written as

$$f'_+ = f^C_+ + f''_+$$

where $f^C_+ \in \mathscr{H}^C_+$ and $f''_+ \in \mathscr{H}''_+$. Therefore

$$\langle f', E_+\theta E_+ f'\rangle_{\mathscr{H}} = \langle f'_+, \theta f'_+\rangle_{\mathscr{H}} = \langle f^C_+, \theta f^C_+\rangle_{\mathscr{H}} + \langle f''_+, \theta f''_+\rangle_{\mathscr{H}}.$$

But f''_+ is of the form $(f''_{4+}, \vec{p}/p_4 f''_{4+})$; thus

$$\begin{aligned}\langle f''_+, \theta f''_+\rangle_{\mathscr{H}} &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{\boldsymbol{p}^2}\left[f''_{4+}(\boldsymbol{p})\theta f''_{4+}(\boldsymbol{p}) + \frac{\vec{p}}{p_4} f''_{4+}(\boldsymbol{p})\theta\left(\frac{\vec{p}}{p_4} f''_{4+}(\boldsymbol{p})\right)\right] \\ &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{\boldsymbol{p}^2}\left(-f''_{4+}(\vec{p}, p_4) f''_{4+}(\vec{p}, -p_4) - \frac{\vec{p}^2}{p_4^2} f''_{4+}(p, p_4) f''_{4+}(p, -p_4)\right) \\ &= -\int_{\mathbb{R}^4_0} \frac{\mathrm{d}^4 p}{\boldsymbol{p}^2}\left(1 + \frac{\vec{p}^2}{p_4^2}\right)|f''_{4+}(\boldsymbol{p})|^2 = 0.\end{aligned}$$

Therefore $E_+\theta E_+ = E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}}$. By noting that f_+^{C} is θ-invariant it is easy to show that $E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}} \geqslant 0$. To show that $E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}}$ is a projector, we need to verify that $(E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}})^2 = E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}}$:

$$\begin{aligned}\langle f_+^{\mathrm{C}}, (E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}})^2 f_+^{\mathrm{C}}\rangle_{\mathscr{K}} &= \langle f_+^{\mathrm{C}}, \theta E_+^{\mathrm{C}}\theta f_+^{\mathrm{C}}\rangle_{\mathscr{K}}\\ &= \langle f_+^{\mathrm{C}}, \theta E_+^{\mathrm{C}} f_+^{\mathrm{C}}\rangle_{\mathscr{K}}\\ &= \langle f_+^{\mathrm{C}}, E_+^{\mathrm{C}}\theta E_+^{\mathrm{C}} f_+^{\mathrm{C}}\rangle_{\mathscr{K}},\end{aligned}$$

where we have again made use of the time-reflection invariance of $\mathscr{K}^{\mathrm{C}}$. Finally, by using an argument similar to that given by Hegerfeldt (1974), it can be shown that $E_+^{\mathrm{C}}E_-^{\mathrm{C}} = E_0^{\mathrm{C}}$, which completes the proof. □

Here we note that $\mathscr{A}_i^{\mathrm{G}}$, the Euclidean vector potential in covariant gauges and defined as a generalised gaussian vector field over $\mathscr{K}$, is Markovian in Nelson's sense. The embedding of $\mathscr{K}$ in $\mathscr{K}^{\mathrm{C}}$ is nonlocal, so it is not surprising that $\mathscr{A}_i^{\mathrm{C}}$ as a generalised vector field over $\mathscr{K}^{\mathrm{C}}$ is only Markovian with respect to special half-spaces.

Furthermore, the T positivity we have shown above is just the Osterwalder–Schrader positivity. However, due to the fact that $\mathscr{A}_i^{\mathrm{G}}$ is non-reflective, one cannot make use of the known result that the Markov path space implies the Osterwalder–Schrader path space (see Klein 1978).

To find the connection between $\mathscr{A}_i^{\mathrm{G}}$ and $\mathscr{A}_i^{\mathrm{C}}$, we shall consider the 'Euclidean one-particle physical space' as $\mathscr{K}^{\mathrm{G}} \equiv \mathscr{K}'/\mathscr{K}''$. Then the Euclidean version of proposition 1 is valid.

Proposition 5. There exists a unitary map

$$\gamma_E : f_i \to f_i - p_i(f_4/p_4)$$

which defines a unitary equivalence $\mathscr{K}^{\mathrm{C}} \cong \mathscr{K}^{\mathrm{G}}$.

The proof is similar to that for proposition 1 and therefore we shall omit it. One can extend this result to the case of Euclidean (or Schwinger) field algebra for photon potentials in a manner similar to that for the relativistic potentials.

To recover the relativistic one-particle space $\mathscr{H}^{\mathrm{C}}$ from $\mathscr{K}^{\mathrm{C}}$ the method of Osterwalder and Schrader (1973, 1975) can be used (see also Ozkaynak 1974). Osterwalder–Schrader positivity enables $\langle \theta f_+^{\mathrm{C}}, g_+^{\mathrm{C}}\rangle_{\mathscr{K}}$ to define a positive sesquilinear form on $\mathscr{K}_+^{\mathrm{C}} \times \mathscr{K}_+^{\mathrm{C}}$. Let N be the closed linear subspace of $\mathscr{K}_+^{\mathrm{C}}$ which consists of vectors h such that $\langle \theta h, h\rangle_{\mathscr{K}} = 0$. Then $\{\mathscr{K}_+^{\mathrm{C}}/N, \langle .,. \rangle_{\mathscr{K}_+^{\mathrm{C}}}\}$ defines a pre-Hilbert space whose closure can be identified with the real relativistic one-particle space $\mathscr{H}_{\mathrm{r}}^{\mathrm{C}}$.

Proposition 6. $\mathscr{H}_{\mathrm{r}}^{\mathrm{C}}$ is isomorphic to the closure of $\mathscr{K}_+^{\mathrm{C}}/N$, with the topology given by $\langle f_+, g_+\rangle_{\mathscr{K}_+^{\mathrm{C}}} = \langle \theta f_+, g_+\rangle_{\mathscr{K}}$.

Finally we shall establish the free Feynman–Kac–Nelson formula for the vector potential in the Coulomb gauge. Let $\mathscr{L}^2(M)$ be the space of square-integrable functions on the sample space of $\mathscr{A}_i^{\mathrm{C}}$, measurable with respect to the σ algebra generated by $\{\mathscr{A}^{\mathrm{C}}(f)) | f \in M\}$. Denote by J the projection of $\mathscr{L}^2(\mathscr{K})$ onto $\mathscr{L}^2(\mathscr{H}_{\mathrm{r}}^{\mathrm{C}})$, and let T_s be the induced unitary action of the time-translation in $\mathscr{K}$ with $T_s f(x, x_4) = f(x, x_4 - s)$. Then the free Feynman–Kac–Nelson formula is given by the following proposition.

Proposition 7. $e^{-sH_0}u = JT_s u$, $u \in \mathscr{L}^2(\mathscr{H}_r^C)$ where H_0 is the free Hamiltonian on $\mathscr{L}^2(\mathscr{H}_r^C)$.

Proof. Since $\mathscr{H}_r^C \subset \mathscr{H}^C$ is closed in $\mathscr{H}^C$ under time translation and complex conjugation, this is in fact a result on $\mathscr{H}^C$ rather than $\mathscr{H}'$. Consider those $f \in \mathscr{H}^C$ which are C^∞ functions with compact support such that $\tilde{f}(\boldsymbol{p}) = 0$ if $|\vec{p}| < \epsilon$ or $|\boldsymbol{p}| > \epsilon^{-1}$ for some $\epsilon > 0$. Such test functions are dense in $\mathscr{H}^C$. If f and g are two such functions, then for real s and t, $f(x) \to f(x, s)$ and $g(x) \to g(x, t)$ are in $\mathscr{H}_r^C$. Then by a direct computation

$$\langle f, g\rangle_{\mathscr{H}} = \sum_i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tfrac{1}{2}\{\exp(-|t-s|.|\vec{p}|)[f_i(\vec{x}, s)g_i(\vec{x}, t)]_{\mathscr{H}_r^C}\}\, dt\, ds.$$

But $\mathscr{H}^C$ is the closure of the space of such functions, so the time translation group in $\mathscr{H}^C$ is the minimal unitary extension of the semi-group in $\mathscr{H}_r^C$. The rest of the proof then follows the same argument as for the scalar case (see, for example, Simon 1974).

4. Conclusion

The Euclidean formulation of free electromagnetic potentials in various gauges has some nice features. For example, in the covariant gauges, the Euclidean potentials do not have difficulties caused by gauge problems. For the nonlocal Coulomb gauge vector potential, the 'Euclideanisation' gives rise to some kind of local structure in terms of the Markov property with respect to special half-spaces. The generalisation of our results to the linearised gravitational potential has been studied by the author (Lim 1979). Recently, the Euclidean method has also been applied to non-Abelian gauge theories such as the Yang–Mills field (see, for example, Osterwalder and Seiler 1978, Schrader 1978).

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